Uncertain nonlinear system control with fuzzy equations and Z-numbers

Raheleh Jafari, Wen Yu*
Departamento de Control Automático
CINVESTAV-IPN (National Polytechnic Institute)
Mexico City, Mexico.

Abstract

Many nonlinear systems can be modeled by uncertain linear-in-parameter models. In this paper, the uncertainty parameters are the $Z$-number coefficients. We use dual fuzzy equations as the models for the uncertain nonlinear systems. The solutions of these fuzzy equations are the controllers when the desired references are regarded as the outputs. The conditions of controllability are proposed. Two types of neural networks are implemented to approximate solutions of the fuzzy equations with $Z$-number coefficients. The proposed methods are validated with some examples.

Keywords: fuzzy equation, Z-number, fuzzy control

1 Introduction

Control of uncertain system is classified in two methodologies: direct and indirect techniques [12]. The methodology involves the direct control incorporates uncertain system as a controlling mechanism, whereas the indirect uncertain model is used to approximate the nonlinear system as a first step, then proceeds controller design based on uncertain model. The indirect fuzzy controller works on the principle of generalized topological structure as well as universal approximation capacity associated to fuzzy model. It has been utilized primarily considering the case of uncertain nonlinear system control. This paper utilizes the indirect control method.

Fuzzy method is a highly favorable tool for uncertain nonlinear system modeling. The fuzzy models approximate uncertain nonlinear systems with several linear piecewise systems.

*The corresponding author is Wen Yu, tel.: +52-55-57473734, email: yuw@ctrl.cinvestav.mx
Mamdani models use fuzzy rules to achieve a good level of approximation of uncertainties [26]. In recent days, many methods involve uncertainties using fuzzy numbers [10][17][32], where the uncertainties of the system are represented by fuzzy coefficients.

The application of the fuzzy equations is in direct connection with the nonlinear control. Given a fuzzy equation, the control incorporated in the equation is in fact a solution of the equation. There are number of techniques to study the solutions of fuzzy equations. [13] used the fuzzy number on parametric shapes and replaced the original fuzzy equations with crisp linear systems. A survey on the extension principle is proposed by [10]. The fuzzy equation holds real or complex coefficients. Nevertheless, there will be no guarantee that the solution exists. [1] proposed the homeotypic analysis technique. [2] used the Newton methodology. In [7] the solution associated to the fuzzy equations is studied by the fixed point technique. One of the most popular methods is the \( \alpha \)-level [15]. By applying the technique of overlay of sets, fuzzy numbers can be resolved [27]. The fuzzy fractional differential and integral equations have been investigated extensively in [3][8][31][36]. Nevertheless, the analytical solutions of fuzzy equations are difficult to obtain and the aforementioned techniques involve greater complexity.

The numerical solution associated to the fuzzy equation and the fuzzy differential equations [25] can be extracted by iterative technique [22], interpolation technique [37] and the Runge-Kutta technique [30]. However, the implementation of these techniques are difficult. Recent results shown that the fusion of the neural networks and the fuzzy logic gives remarkable success in nonlinear system modeling [38]. The neural networks may also be used to solve fuzzy equations. [9] uses a neural network with three neurons to estimate the second degree fuzzy equation. [19] and [21] extend the result of [9] to fuzzy polynomial equations. [29] gives a matrix form of the neuronal learning.

The Z-number is a novel idea that is subjected to a higher potential in order to illustrate the information of the human being as well as to use in information processing [39]. Z-numbers can be regarded as to answer questions and carry out the decisions [23]. There are few structure based on the theoretical concept of Z-numbers [14]. [4] gives an inception which results in the extension of the Z-numbers. [24] proposes a theorem to transfer the Z-numbers to the usual fuzzy sets. Many fields related to the analysis of the decisions use the ideas of Z-numbers. Z-numbers are much less complex to calculate when compared to nonlinear system modeling methods. The Z-number is abundantly adequate number than the fuzzy number. Although Z-numbers are implemented in many literatures, from theoretical point of view this approach is not certified completely.

In this paper, we use dual fuzzy equations [37] to model the uncertain nonlinear systems, where the coefficients are Z-numbers and the Z-numbers are on both sides of the equation. Then we discuss the existence of the solutions of the dual fuzzy equations. It corresponds to
controllability problem of the fuzzy control [11]. Finally, we use two types of neural networks, feed-forward and feedback networks, to approximate the solutions (control actions) of the dual fuzzy equation. Several examples are utilized in order to demonstrate the effectivity of our fuzzy control design methods.

2 Nonlinear system modeling with dual fuzzy equations and Z-numbers

A general discrete-time nonlinear system can be described as

\[ x_{k+1} = \bar{f} [x_k, u_k], \quad y_k = \bar{g} [x_k] \]  \hspace{1cm} (1)

Here we consider \( u_k \in \mathbb{R}^a \) as the input vector, \( x_k \in \mathbb{R}^l \) is regarded as an internal state vector and \( y_k \in \mathbb{R}^m \) is the output vector. \( \bar{f} \) and \( \bar{g} \) are noted as generalized nonlinear smooth functions \( \bar{f}, \bar{g} \in C^\infty \). Define \( Y_k = [y_{k+1}^T, y_k^T, \cdots]^T \) and \( U_k = [u_{k+1}^T, u_k^T, \cdots]^T \). Suppose \( \frac{\partial Y}{\partial x} \) is non-singular at the instance \( \bar{x}_k = 0, U_k = 0 \), this will create a path towards the following model

\[ y_k = \Psi [y_{k-1}^T, y_{k-2}^T, \cdots, u_k^T, u_{k-1}^T, \cdots] \] \hspace{1cm} (2)

where \( \Psi (\cdot) \) is an nonlinear difference equation exhibiting the plant dynamics, \( u_k \) and \( y_k \) are computable scalar input and output respectively, \( d \) is noted to be time delay. The nonlinear system which is represented by (2) is implied as a NARMA model. The input of the system with incorporated nonlinearity is considered to be as

\[ x_k = [y_{k-1}^T, y_{k-2}^T, \cdots, u_k^T, u_{k-1}^T, \cdots]^T \]

Taking into consideration the nonlinear systems as mentioned in (2), it can be simplified as the following linear-in-parameter model

\[ z_k = \sum_{i=1}^{n} a_i f_i (x_k) \] \hspace{1cm} (3)

or

\[ z_k + \sum_{i=1}^{m} b_i g_i (x_k) = \sum_{i=1}^{n} a_i f_i (x_k) \] \hspace{1cm} (4)

here \( a_i \) and \( b_i \) are considered to be the linear parameters, \( f_i (x_k) \) and \( g_i (x_k) \) are nonlinear functions. The variables related to these functions are quantifying input and output. A popular example of this pattern of model is considered to be a robot manipulator [33]

\[ M (p) \ddot{p} + C (p, \dot{p}) \dot{p} + B \dot{p} + g (p) = \tau \] \hspace{1cm} (5)
can be explained as
\[ \sum_{i=1}^{n} Y_i(p, \bar{p}, \tilde{p}) \theta_i = \tau \]  

The modeling of uncertain nonlinear systems can be achieved by utilizing the linear-in-parameter models linked to fuzzy parameters. We assume the model of the nonlinear systems (3) and (4) have uncertainties in the parameters \( a_i \) and \( b_i \). These uncertainties are in the sense of Z-numbers [40]. The following definitions will be used in this paper.

**Definition 1 (fuzzy number)** A fuzzy number \( A \) is a function \( A : \mathbb{R} \rightarrow [0, 1] \), in such a way, 1) \( A \) is normal, (there prevail \( x_0 \in \mathbb{R} \) in such a way \( A(x_0) = 1 \); 2) \( A \) is convex, \( A(\lambda x + (1 - \lambda)y) \geq \min\{A(x), A(y)\}, \forall x, y \in \mathbb{R}, \forall \lambda \in [0, 1] \); 3) \( A \) is upper semi-continuous on \( \mathbb{R} \), i.e., \( A(x) \leq A(x_0) + \varepsilon, \forall x \in N(x_0), \forall x_0 \in \mathbb{R}, \forall \varepsilon > 0, N(x_0) \) is a neighborhood; 4) The set \( A^+ = \{x \in \mathbb{R}, A(x) > 0\} \) is compact.

**Definition 2 (Z-numbers)** A Z-number has two components \( Z = [A(x), p] \). The primary component \( A(x) \) is termed as a restriction on a real-valued uncertain variable \( x \). The secondary component \( p \) is a measure of reliability of \( A \). \( p \) can be reliability, strength of belief, probability or possibility. When \( A(x) \) is a fuzzy number and \( p \) is the probability distribution of \( x \), the Z-number is defined as \( Z^+ \)-number. When both \( A(x) \) and \( p \) are fuzzy numbers, the Z-number is defined as \( Z^- \)-number.

The \( Z^+ \)-number carries more information than the \( Z^- \)-number. In this paper, we use the definition of \( Z^+ \)-number, i.e., \( Z = [A, p] \), \( A \) is a fuzzy number, \( p \) is a probability distribution.

In order to demonstrate the fuzzy numbers, the membership functions are utilized. The most widely discussed membership functions are noted to be the triangular function
\[
\mu_A = F(a, b, c) = \begin{cases} \frac{x-a}{b-a} & a \leq x \leq b \\ \frac{c-x}{c-b} & b \leq x \leq c \\ 0 & \text{otherwise} \end{cases} \]  

as well as trapezoidal function
\[
\mu_A = F(a, b, c, d) = \begin{cases} \frac{x-a}{b-a} & a \leq x \leq b \\ \frac{d-x}{d-c} & c \leq x \leq d \\ 0 & \text{otherwise} \end{cases} \]  

The probability measure is expressed as
\[ P = \int_R \mu_A(x)p(x)dx \]  
where \( p \) is the probability density of \( x \) and \( R \) is the restriction on \( p \). For discrete Z-numbers, we have
\[ P(A) = \sum_{i=1}^{n} \mu_A(x_i)p(x_i) \]
The space of discrete fuzzy sets is denoted by $E$. $E_{[a,b]}$ denotes the space of discrete fuzzy sets of $[a, b] \subset R$. Signifying $\tilde{Z}$ the space of discrete Z-numbers as

$$\tilde{Z} = \{Z = (A, p) | A \in \tilde{E}, p \in \tilde{E}_{[0,1]}\}$$

(11)

**Definition 3 (α-level of fuzzy number)** The α-level associated to a fuzzy number $A$ is stated as

$$[A]^{\alpha} = \{x \in \Re : A(x) \geq \alpha\}$$

(12)

also, $0 < \alpha \leq 1$. Or

$$[A]^{\alpha} = (A^{\alpha}, \overline{A}^{\alpha})$$

In order to operate the Z-number, we propose the following definition.

**Definition 4 (α-level of Z-numbers)** The α-level of the Z-number $Z = (A, p)$ is demonstrated as

$$[Z]^{\alpha} = ([A]^{\alpha}, [p]^{\alpha})$$

(13)

where $0 < \alpha \leq 1$. $[p]^{\alpha}$ is calculated by the Nguyen’s theorem

$$[p]^{\alpha} = p([A]^{\alpha}) = p((A^{\alpha}, \overline{A}^{\alpha})) = [P^{\alpha}, \overline{P}^{\alpha}]$$

where $p([A]^{\alpha}) = \{p(x) | x \in [A]^{\alpha}\}$. So $[Z]^{\alpha}$ can be expressed as the form α-level of a fuzzy number

$$[Z]^{\alpha} = (Z^{\alpha}, \overline{Z}^{\alpha}) = ((A^{\alpha}, \overline{A}^{\alpha}), (P^{\alpha}, \overline{P}^{\alpha}))$$

(14)

where $P^{\alpha} = A^{\alpha} p(x^{\alpha}_i)$, $\overline{P}^{\alpha} = \overline{A}^{\alpha} p(\overline{x}^{\alpha}_i)$, $[x_i]^{\alpha} = (x^{\alpha}_i, \overline{x}^{\alpha}_i)$.

Similarly with the fuzzy numbers [17], the Z-numbers are also incorporated with four primary operations: $\oplus$, $\ominus$, $\odot$ and $\oslash$. These operations are exhibited by: sum, subtract, multiply and division. The operations in this paper are different from that mentioned in [40]. The α-level of Z-numbers is applied to simplify the operations.

Let us consider $Z_1 = (A_1, p_1)$ and $Z_2 = (A_2, p_2)$ be two discrete Z-numbers illustrating the uncertain variables $x_1$ and $x_2$, also $\sum_{k=1}^{n} p_1(x_{1k}) = 1$ and $\sum_{k=1}^{n} p_2(x_{2k}) = 1$. The operations are defined as

$$Z_{12} = Z_1 * Z_2 = (A_1 * A_2, p_1 * p_2)$$

where $* \in \{\oplus, \ominus, \odot, \oslash\}$.

The operations for the fuzzy numbers are defined as [17]

$$[A_1 \oplus A_2]^{\alpha} = [A_1^{\alpha}, A_2^{\alpha}, \overline{A}_1^{\alpha}, \overline{A}_2^{\alpha}]$$

$$[A_1 \ominus A_2]^{\alpha} = [A_1^{\alpha} - A_2^{\alpha}, \overline{A}_1^{\alpha} - \overline{A}_2^{\alpha}]$$

$$[A_1 \odot A_2]^{\alpha} = (A_1^{\alpha} A_2^{\alpha} - A_1^{\alpha} A_2^{\alpha}, \overline{A}_1^{\alpha} \overline{A}_2^{\alpha} - \overline{A}_1^{\alpha} \overline{A}_2^{\alpha})$$

(15)
For all \( p_1 \ast p_2 \) operations, we use convolutions for the discrete probability distributions

\[
p_1 \ast p_2 = \sum_i p_1(x_{1,i})p_2(x_{2,(n-i)}) = p_{12}(x)
\]

The above definitions satisfy the Hukuhara difference [5]

\[
Z_1 \ominus_H Z_2 = Z_{12} \\
Z_1 = Z_2 \oplus Z_{12}
\]

Here if \( Z_1 \ominus_H Z_2 \) prevail, the \( \alpha \)-level is

\[
[Z_1 \ominus_H Z_2]^\alpha = [Z_1^\alpha - Z_2^\alpha, \overline{Z}_1^\alpha - \overline{Z}_2^\alpha]
\]

Obviously, \( Z_1 \ominus_H Z_1 = 0 \), \( Z_1 \ominus Z_1 \neq 0 \).

If \( A \) is a triangle function, the absolute value of the Z-number \( Z = (A, p) \) is

\[
|Z(x)| = (|a_1| + |b_1| + |c_1|, p(|a_2| + |b_2| + |c_2|))
\]

(16)

Now we utilize fuzzy equations (3) or (4) to model the uncertain nonlinear system (2). The parameters of the fuzzy equations (3) or (4) are in the form of Z-numbers

\[
y_k = a_1 \odot f_1(x_k) \oplus a_2 \odot f_2(x_k) \oplus \ldots \oplus a_n \odot f_n(x_k)
\]

(17)

or

\[
a_1 \odot f_1(x_k) \oplus a_2 \odot f_2(x_k) \oplus \ldots \oplus a_n \odot f_n(x_k)
= b_1 \odot g_1(x_k) \oplus b_2 \odot g_2(x_k) \oplus \ldots \oplus b_m \odot g_m(x_k) \oplus y_k
\]

(18)

where \( a_i \) and \( b_i \) are Z-numbers. (18) is considered to be more general as compared to (17), it is termed as dual fuzzy equation.

Taking into consideration a particular case, \( f_i(x_k) \) has polynomial pattern,

\[
(a_1 \odot x_k) \oplus \ldots \oplus (a_n \odot x_k^n) = (b_1 \odot x_k) \oplus \ldots \oplus (b_n \odot x_k^n) \oplus y_k
\]

(19)

(19) is termed as dual polynomial based on Z-number.

The main intention associated with the modeling is to diminish error in midst of two output \( y_k \) and \( z_k \). As \( y_k \) is noted as a Z-number and \( z_k \) is considered to be crisp Z-number, hence we apply the minimum of every points as the model mentioned below

\[
\max_k |y_k - z_k| = \max_k |\beta_k|
\]

\[
y_k = ((u_1(k), u_2(k), u_3(k)), p(v_1(k), v_2(k), v_3(k)))
\]

\[
\beta_k = ((\rho_1(k), \rho_2(k), \rho_3(k)), p(\varphi_1(k), \varphi_2(k), \varphi_3(k)))
\]

(20)
By the definition of absolute value (16), we conclude

\[
\max_k |\beta_k| = \max_k [(|u_1(k) - f(x_k)| + |u_2(k) - f(x_k)| + |u_3(k) - f(x_k)|, (\rho_1(v_1(k)) - f(x_k)) + |p(v_2(k)) - f(x_k)| + |p(v_3(k)) - f(x_k)|)]
\]

\[
 \rho_1(k) = \max_k |u_1(k) - f(x_k)|, \quad \rho_2(k) = \max_k |u_2(k) - f(x_k)|, \quad \rho_3(k) = \max_k |u_3(k) - f(x_k)|
\]

\[
p(\varphi_1(k)) = \max_k |p(v_1(k)) - f(x_k)|, \quad p(\varphi_2(k)) = \max_k |p(v_2(k)) - f(x_k)|, \quad p(\varphi_3(k)) = \max_k |p(v_3(k)) - f(x_k)|
\]

(21)

The modelling constraint (20) is to uncover \(u_1(k), u_2(k), u_3(k), p(v_1(k)), p(v_2(k))\) and \(p(v_3(k))\) in such a manner

\[
\min_{u_i(k), p(v_i(k))} \left\{ \max_k |\beta_k| \right\} = \min_{u_i(k), p(v_i(k))} \left\{ \max_k |y_k - f(x_k)| \right\}, \quad i = 1, 2, 3 \tag{22}
\]

Considering (21), we have

\[
\rho_1(k) \geq |u_1(k) - f(x_k)|, \quad \rho_2(k) \geq |u_2(k) - f(x_k)|, \quad \rho_3(k) \geq |u_3(k) - f(x_k)|
\]

\[
p(\varphi_1(k)) \geq |p(v_1(k)) - f(x_k)|, \quad p(\varphi_2(k)) \geq |p(v_2(k)) - f(x_k)|, \quad p(\varphi_3(k)) \geq |p(v_3(k)) - f(x_k)|
\]

(22) can be resolved by the application of linear programming methodology

\[
\begin{align*}
\min \rho_1(k) \\
\text{subject: } & \rho_1(k) + \{ \sum_{j=0}^{n} a_j \odot x_j^i \} \cap H (\sum_{j=0}^{n} b_j \odot x_j^i) \geq f(x_k) \\
& \rho_1(k) - \{ \sum_{j=0}^{n} a_j \odot x_j^i \} \cap H (\sum_{j=0}^{n} b_j \odot x_j^i) \geq -f(x_k)
\end{align*} \tag{23}
\]

\[
\min \varphi_1(k) \\
\text{subject: } & p(\varphi_1(k)) + \{ \sum_{j=0}^{n} a_j \odot x_j^i \} \cap H (\sum_{j=0}^{n} b_j \odot x_j^i) \geq f(x_k) \\
& p(\varphi_1(k)) - \{ \sum_{j=0}^{n} a_j \odot x_j^i \} \cap H (\sum_{j=0}^{n} b_j \odot x_j^i) \geq -f(x_k)
\]

\[
\begin{align*}
\min \rho_2(k) \\
\text{subject: } & \rho_2(k) - \left[ \sum_{j=0}^{n} a_j \odot x_j^i - \sum_{j=0}^{n} b_j \odot x_j^i \right] \geq f(x_k) \\
& \rho_2(k) \geq 0
\end{align*} \tag{24}
\]

\[
\min \varphi_2(k) \\
\text{subject: } & p(\varphi_2(k)) - \left[ \sum_{j=0}^{n} a_j \odot x_j^i - \sum_{j=0}^{n} b_j \odot x_j^i \right] \geq f(x_k) \\
& p(\varphi_2(k)) \geq 0
\]

\[
\begin{align*}
\min \rho_3(k) \\
\text{subject: } & \rho_3(k) - \left[ \sum_{j=0}^{n} \bar{a}_j \odot \bar{x}_j^i - \sum_{j=0}^{n} \bar{b}_j \odot \bar{x}_j^i \right] \geq f(x_k) \\
& \rho_3(k) \geq 0
\end{align*} \tag{25}
\]

\[
\min \varphi_3(k) \\
\text{subject: } & p(\varphi_3(k)) - \left[ \sum_{j=0}^{n} \bar{a}_j \odot \bar{x}_j^i - \sum_{j=0}^{n} \bar{b}_j \odot \bar{x}_j^i \right] \geq f(x_k) \\
& p(\varphi_3(k)) \geq 0
\]
here \(a_j, b_j, x_k, \bar{a}_j, \bar{b}_j\) and \(\bar{x}_k\) are explained as mentioned in (13). Henceforth, the superior way of approximating \(f(x_k)\) at the juncture \(x_k\) is \(y_k\). The minimization of the approximation error which is termed as \(\beta_k\) is achieved.

The process involved in order to design the controller is to obtain \(u_k\), in such a manner that the output related to the plant \(y_k\) can approach to the desired output \(y_k^*\), or the trajectory tracking error diminishes

\[
\min_{u_k} \|y_k - y_k^*\| \quad (26)
\]

This control entity can be regarded as to detect a solution \(u_k\) for the following dual equation on the basis of Z-number

\[
(a_1 \odot f_1(x_k)) \oplus (a_2 \odot f_2(x_k)) \oplus \ldots \oplus (a_n \odot f_n(x_k)) = (b_1 \odot g_1(x_k)) \oplus (b_2 \odot g_2(x_k)) \oplus \ldots \oplus (b_m \odot g_m(x_k)) \oplus y_k^* \quad (27)
\]

where \(x_k = [y_k^T, y_{k-1}^T, \ldots, u_k^T, u_{k-1}^T, \ldots]^T\)

### 3 Controllability of uncertain nonlinear systems via dual fuzzy equations and Z-numbers

As the primary concern of control is the finding out of \(u_k\) as mentioned in (18) which is relied on Z-number, the controllability constraint signifies that the dual fuzzy equation (18) involves solution.

We need the following lemmas:

**Lemma 1** If the coefficients of the dual equation (18) are Z-numbers, then the solution \(u_k\) satisfies

\[
\{ \cap_{j=1}^n \text{domain } [f_j(x)] \} \cap \{ \cap_{j=1}^m \text{domain } [g_j(x)] \} \neq \phi \quad (28)
\]

**Proof.** Assume \(u_0 \in \widehat{Z}\) is considered to be a solution of (18), the dual equation which relies on Z-numbers turns out to be

\[
(a_1 \odot f_1(u_0)) \oplus \ldots \oplus (a_n \odot f_n(u_0)) = (b_1 \odot g_1(u_0)) \oplus \ldots \oplus (b_m \odot g_m(u_0)) \oplus y_k^*
\]

As \(f_j(u_0)\) and \(g_j(u_0)\) prevails, \(u_0 \in \text{domain}[f_j(x)]\), \(u_0 \in \text{domain}[g_j(x)]\). Subsequently, it can be inferred that \(u_0 \in \cap_{j=1}^n \text{domain } [f_j(x)] = C_1\) and \(u_0 \in \cap_{j=1}^m \text{domain } [g_j(x)] = C_2\). Hence there prevail \(u_0\), in such a manner \(u_0 \in C_1 \cap C_2 \neq \phi\).

Let two Z-numbers \(p_0, q_0 \in \widehat{Z}, p_0 < q_0\). We define a set \(D(x) = \{x \in \widehat{Z}, p_0 \leq x \leq q_0\}\) and an operator \(W : D \rightarrow D\) as

\[
W(p_0) \geq p_0, \quad W(q_0) \leq q_0 \quad (29)
\]
In this matter, $W$ is condensing and continuous, also it is bounded as $W(z) < r(z)$, $z \in D$ and $r(z) > 0$. $r(z)$ can be considered as the evaluation of $z$.

**Lemma 2** We define $q_i = W(q_{i-1})$ and $p_i = W(p_{i-1})$, $i = 1, 2, \ldots$, and the upper and lower bounds of $W$ are $\bar{w}$ and $\underline{w}$, then

$$
\bar{w} = \lim_{i \rightarrow +\infty} q_i, \quad \underline{w} = \lim_{i \rightarrow +\infty} p_i,
$$

and

$$
p_0 \leq p_1 \leq \ldots \leq p_n \leq \ldots \leq q_n \leq \ldots \leq q_1 \leq q_0.
$$

**Proof.** As long as $W$ is uprising, it is quite obvious from (29) that (31) prevail. In this case we verify that $\{p_i\}$ conjoins to some $\underline{w} \in \hat{Z}$ and $W(\underline{w}) = \underline{w}$. The set $B = \{p_0, p_1, p_2, \ldots\}$ is enclosed and $B = W(B) \cup \{p_0\}$, thus $r(B) = r(W(B))$, here $r(B)$ denotes the quantification of non-compactness of $B$. It is observed from $W$ that $r(B) = 0$, i.e., $B$ is a proportionally compact set. Thus, there prevail an outflow of $\{p_{n_k}\} \subset \{p_n\}$ in such a manner that $p_{n_k} \rightarrow \underline{w}$ for any $\underline{w} \in \hat{Z}$ (take into consideration that $\hat{Z}$ is complete). Distinctly, $p_n \leq \underline{w} \leq q_n$ $(n = 1, 2, \ldots)$. As in case $m > n_k$, according to [6], the supremum metrics $D(\underline{w}, p_m) \leq D(\underline{w}, p_{n_k})$. Hence, $p_m \rightarrow \underline{w}$ as $m \rightarrow \infty$. Considering limit $n \rightarrow \infty$ on either sides of the equality $p_n = W(p_{n-1})$, we find $w = W(w)$, as a result $W$ is continuous and $D$ is closed.

Similarly, we can conclude that $\{q_n\}$ converges to some $\bar{w} \in \hat{Z}$ and $W(\bar{w}) = \bar{w}$. So we confirm that $\bar{w}$ and $\underline{w}$ are the maximal and minimal fixed point related to $W$ in $D$ respectively. Assume $\bar{w} \in D$ and $W(\bar{w}) = \bar{w}$. As $W$ is in the increasing tend, it is obvious from $p_0 \leq \bar{w} \leq q_0$ that $W(p_0) \leq W(\bar{w}) \leq W(q_0)$, i.e., $p_1 \leq \bar{w} \leq q_1$. Utilizing the similar logic, we obtain $p_2 \leq \bar{w} \leq q_2$, and formally, $p_n \leq \bar{w} \leq q_n$ $(n = 1, 2, 3, \ldots)$. Here, considering limit $n \rightarrow \infty$, we extract $\bar{w} \leq \underline{w} \leq \bar{w}$.

The fixed point will result in $x_0$ inside $D$, the consecutive iterates $x_i = W(x_{i-1})$, $i = 1, 2, \ldots$ will result in convergency towards $x_0$, i.e., $\lim_{i \rightarrow \infty} D(x_i, x_0) = 0$. ■

**Theorem 1** Let us consider $Z = (Z^\alpha, \overline{Z}^\alpha)$, where $Z^\alpha = (d_{M_1}(\alpha), d_{M_2}(\alpha))$, $\overline{Z}^\alpha = (d_{U_1}(\alpha), d_{U_2}(\alpha))$, $\alpha \in [0, 1]$. If $a_i$ and $b_j$ $(i = 1 \cdots n, j = 1 \cdots m)$ in (18) are Z-numbers and they suffice the Lipschitz condition

$$
|d_{M_1}(a_i) - d_{M_2}(a_i)| - |d_{M_1}(a_k) - d_{M_2}(a_k)| \leq H |a_i(M_1) - a_k(M_1)| + H |a_i(M_2) - a_k(M_2)|
$$

$$
|d_{U_1}(a_i) - d_{U_2}(a_i)| - |d_{U_1}(a_k) - d_{U_2}(a_k)| \leq H |a_i(U_1) - a_k(U_1)| + H |a_i(U_2) - a_k(U_2)|
$$

also, the upper bounds of the functions $f_i$ and $g_j$ are $|f_i| \leq \overline{f}$, $|g_j| \leq \overline{g}$, then the dual fuzzy equation (18) has a solution $u$ in the set mentioned below

$$
K_H = \left\{ u \in \overline{Z} : |\overline{\pi}^{\alpha_1, \beta_1} - \overline{\pi}^{\alpha_2, \beta_2}| \leq (n\overline{f} \oplus \overline{m}\overline{g})(H |\alpha_1 - \alpha_2| + H |\beta_1 - \beta_2|) \right\}
$$

(33)
Proof. Since \( a_i \) and \( b_j \) are Z-numbers and from (32) we have

\[
d_M(\alpha, \beta) = ((a_{1M_1}(\alpha), a_{1M_2}(\beta)) \odot f_1(x)) \odot \cdots \odot ((a_{nM_1}(\alpha), a_{nM_2}(\beta)) \odot f_n(x))
\]
\[
\odot_H((b_{1M_1}(\alpha), b_{1M_2}(\beta)) \odot g_1(x)) \odot_H \cdots \odot_H ((b_{mM_1}(\alpha), b_{mM_2}(\beta)) \odot g_m(x))
\]

Hence

\[
|d_M(\alpha, \beta) - d_M(\varphi, \rho)| = (|f_1(x)| \odot |(a_{1M_1}(\alpha), a_{1M_2}(\beta)) \odot_H (a_{1M_1}(\varphi), a_{1M_2}(\rho))|) \odot \cdots \odot (|f_n(x)| \odot |(a_{nM_1}(\alpha), a_{nM_2}(\beta)) \odot_H (a_{nM_1}(\varphi), a_{nM_2}(\rho))|) \odot (|g_1(x)| \odot |(b_{1M_1}(\alpha), b_{1M_2}(\beta)) \odot_H (b_{1M_1}(\varphi), b_{1M_2}(\rho))|) \odot \cdots \odot (|g_m(x)| \odot |(b_{mM_1}(\alpha), b_{mM_2}(\beta)) \odot_H (b_{mM_1}(\varphi), b_{mM_2}(\rho))|)
\]

(34)

With respect to the Lipschitz condition (32), (34) is

\[
|d_M(\alpha, \beta) - d_M(\varphi, \rho)| \leq \bar{f}(H \sum_{i=1}^{n} |\alpha - \varphi| + H \sum_{i=1}^{n} |\beta - \rho|) + \bar{g}(H \sum_{i=1}^{m} |\alpha - \varphi| + H \sum_{i=1}^{m} |\beta - \rho|)
\]

In the same manner, the upper limits suffice

\[
|d_U(\alpha, \beta) - d_U(\varphi, \rho)| \leq (n\bar{f} + m\bar{g})(H |\alpha - \varphi| + H |\beta - \rho|)
\]

As the lower limit \( |d_M(\alpha, \beta) - d_M(\varphi, \rho)| \geq 0 \), with respect to Lemma 2 the solution contains in \( K_H \) and is defined in (33).

Lemma 3 Let us consider the data number to be \( m \) and also we suggest the order of the equation to be \( n \) in (19) that suffices

\[
m \geq 2n + 1
\]

(35)

considering \( k = 1 \cdots m \), hence the solutions of (24) and (25) are \( \rho_2(k) = p(\varphi_2(k)) = \rho_3(k) = p(\varphi_3(k)) = 0 \).

Proof. Since

\[
\sum_{i=0}^{n} a_i x_k^i - \sum_{j=0}^{m} b_j x_k^j \leq -f(x_k)
\]

Let us opt \( 2n + 1 \) points through \((x_k, f(x_k))\) and find the following interpolation dual polynomial based on Z-number on these data

\[
h(k) = \sum_{i=0}^{n} a_i x_k^i - \sum_{j=0}^{m} b_j x_k^j
\]

(37)

Let \( j = \max \{ h(k) + f(x_k) \} \) and \( j > 0 \), as a result we can transform the dual polynomial (19) to the other form of new dual polynomial \( h(k) - j \). This suggested recent dual polynomial
based on Z-number suffices (36). Since the presumable spot of (24) are \( \rho_2(k) \geq 0 \) and \( p(\varphi_2(k)) \geq 0 \), so it should be zero. In the similar manner, outcome can be extracted for (25).

\[ x_k \text{ and } f(x_k) \text{ are crisp Z-numbers. In case of } k = 1 \cdots n, \text{ there should be a validated solution for the equation approximation [28].} \]

**Theorem 2** If there is a big amount of data number as (35) and the dual polynomial based on Z-number (19) satisfies

\[
D[j(x_{k1}, u_{k1}), j(x_{k2}, u_{k2})] \leq rD[u_{k1}, u_{k2}] \quad 0 < r < 1
\]

where \( j(\cdot) \) exhibits a dual polynomial based on Z-number,

\[
j(x_{k1}, u_{k1}) : (a_1 \odot x_{k1}) \oplus \ldots \oplus (a_n \odot x_{k1}^n) = (b_1 \odot x_{k1}) \oplus \ldots \oplus (b_n \odot x_{k1}^n) \oplus y_{k1}
\]

\( D[u, v] \) is the Hausdorff distance related to Z-numbers \( u \) and \( v \),

\[
D[u, v] = \max \left\{ \sup_{(x_1, y_1) \in u} \inf_{(x_2, y_2) \in v} (d(x_1, x_2) + d(y_1, y_2)), \sup_{(x_1, y_1) \in v} \inf_{(x_2, y_2) \in u} (d(x_1, x_2) + d(y_1, y_2)) \right\}
\]

\( d(x, y) \) is the supremum metrics considering fuzzy sets, then (19) contains a distinct solution \( u \).

**Proof.** According to lemma 2, there exist solutions for (23)-(25), if there are numerous data which satisfy (35). Neglecting deficit of generality, let we consider the solutions for (23)-(25) are at par with \( x_k = 0 \), which tends to \( u_0 \). (38) signifies \( j(\cdot) \) in (39) is continuous. If we select \( \delta > 0 \) in such a manner that \( D[y_{k}, u_0] \leq \delta \), hence

\[
D[j(x_k, u_0), u_0] \leq (1 - r)\delta
\]

where \( j(0, u_0) = u_0 \). Taking into our account we choose \( x \) close to 0, \( x_k \in [0, s], s > 0 \), so

\[
S_0 : \vartheta = \sup_{x_k \in [0, s]} D[y_{k1}, y_{k2}]
\]

Assume \( \{y_{km}\} \) be a succession in \( S_0 \), for any \( \varepsilon > 0 \), the computation of \( N_0(\varepsilon) \) can be done in such a manner \( \vartheta < \varepsilon, m, n \geq N_0 \). Hence \( y_{km} \longrightarrow y_{k} \) for \( x_k \in [0, s] \). Henceforth

\[
D[y_{k}, u_0] \leq D[y_{k}, y_{km}] + D[y_{km}, u_0] < \varepsilon + \delta
\]

for all \( x \in [0, s], m \geq N_0(\varepsilon) \). As \( \varepsilon > 0 \) is randomly minute,

\[
D[y_{k}, u_0] \leq \delta
\]
Figure 1: A feed-forward neural network (NN) approximates the solutions of fuzzy equation for all \( x \in [0, s] \). Now we validate that \( y_k \) is continuous at \( x_0 = 0 \). Taking into consideration \( \delta > 0 \), there prevails \( \delta_1 > 0 \) in such a manner

\[
D[y_k, u_0] \leq D[y_k, y_{km}] + D[y_{km}, u_0] \leq \varepsilon + \delta_1
\]

for every \( m \geq N_0(\varepsilon) \), by means of (41), while \( |x - x_0| < \delta_1 \), \( y_k \) is continuous at \( x_0 = 0 \). As a result (19) contains a distinct solution \( u_0 \).

The necessary circumstance in order to establish the controllability (existence of solution) related to the dual equation (27) is (28), the sufficient condition related to the controllability is (32). For majority of membership functions, such as triangular functions and the trapezoidal function, the Lipschitz condition (32) is fulfilled. In this case it is considered to be controllable.

4 Utilization of neural networks for fuzzy controller design

It is not possible to acquire an analytical solution for (27). Here, neural networks are utilize to approximate the solution (control). In order to fit the neural networks, (27) is written as

\[
(a_1 \circ f_1(x)) \oplus \ldots \oplus (a_n \circ f_n(x)) \oplus_H (b_1 \circ g_1(x)) \oplus_H \ldots \oplus_H (b_m \circ g_m(x)) = y_k^* \tag{42}
\]

We use two types of neural networks, feed-forward and feedback neural networks to approximate the solution of (42), see Figure 1 and Figure 2. The Z-numbers \( a_i \) and \( b_i \) represents the inputs of the neural network, the Z-number \( y_k \) represent the output. \( f_i(x) \) and \( g_j(x) \) are the Z-number weights.

The main idea is to detect appropriate weights of neural networks in such a manner that the output of the neural network \( y_k \), approaches the desired output \( y_k^* \). From the view point
Figure 2: A feedback neural network (FNN) approximates the solutions of fuzzy equation of control, it is utter necessity to find out a suitable controller \( u_k \) which is a function of \( x \), in such a manner that the plant (1) \( y_k \) (crisp value) estimate the Z-number \( y_k^* \).

In the control point of view, we want to find a controller \( u_k \) which is a function of \( x \), such that the output of the plant (1) \( y_k \) (crisp value) approximate the Z-number \( y_k^* \).

The input Z-numbers \( a_i \) and \( b_i \) are primarily implemented to \( \alpha \)-level as (13)

\[
[a_i]^{\alpha} = (a_i^\alpha, \overline{a_i}^\alpha) \quad i = 1 \cdots n
\]

\[
[b_j]^{\alpha} = (b_j^\alpha, \overline{b_j}^\alpha) \quad j = 1 \cdots m
\]

The next step is initiated by multiplying the above relations with the Z-number weights \( f_i(x) \) and \( g_j(x) \) and summarized as

\[
[O_f]^{\alpha} = \left( \sum_{i=M_f} f_i^{\alpha}(x) a_i^{\alpha} + \sum_{i=C_f} f_i^{\alpha}(x) \overline{a_i}^{\alpha}, \sum_{i=M_{f_i}} f_i^{\alpha}(x) \overline{a_i}^{\alpha}, \sum_{i=C_{f_i}} f_i^{\alpha}(x) a_i^{\alpha} \right) \quad (44)
\]

\[
[O_g]^{\alpha} = \left( \sum_{j=M_g} g_j^{\alpha}(x) b_j^{\alpha} + \sum_{j=C_g} g_j^{\alpha}(x) \overline{b_j}^{\alpha}, \sum_{j=M_{g_j}} g_j^{\alpha}(x) \overline{b_j}^{\alpha}, \sum_{j=C_{g_j}} g_j^{\alpha}(x) b_j^{\alpha} \right)
\]

Here \( M_f = \{ i | f_i^{\alpha}(x) \geq 0 \}, C_f = \{ i | f_i^{\alpha}(x) < 0 \}, M_{f_i} = \{ i | \overline{f_i}^{\alpha}(x) \geq 0 \}, C_{f_i} = \{ i | \overline{f_i}^{\alpha}(x) < 0 \}, M_g = \{ j | g_j^{\alpha}(x) \geq 0 \}, C_g = \{ j | g_j^{\alpha}(x) < 0 \}, M_{g_j} = \{ j | \overline{g_j}^{\alpha}(x) \geq 0 \}, C_{g_j} = \{ j | \overline{g_j}^{\alpha}(x) < 0 \}.\)

The neural network output is

\[
[y_k]^{\alpha} = ([O_f]^{\alpha} - [O_g]^{\alpha}, \overline{[O_f]^{\alpha}} - \overline{[O_g]^{\alpha}})
\]

The error of the training is

\[
e_k = y_k^\alpha \ominus \hat{y}_k
\]

here \( [y_k^*]^{\alpha} = (y_k^{\alpha*}, \overline{y_k}^{\alpha*}) \), \( [\hat{y}_k]^{\alpha} = (\hat{y}_k^{\alpha}, \overline{\hat{y}_k}^{\alpha}) \), \( [e_k]^{\alpha} = (e_k^{\alpha}, \overline{e_k}^{\alpha}) \).

A cost function, which is generated on the basis of Z-numbers is implemented for the training of the weights as mentioned below

\[
J_k = J^\alpha + \overline{J}^\alpha, \quad J^\alpha = \frac{1}{2} \left( y_k^{\alpha*} - \hat{y}_k^{\alpha} \right)^2, \quad \overline{J}^\alpha = \frac{1}{2} \left( \overline{y}_k^\alpha - \overline{\hat{y}}_k^\alpha \right)^2
\]

(46)
It is quite obvious, $J_k \to 0$ when $[\hat{y}_k]^\alpha \to [y_k^*]^\alpha$.

The vital positiveness lies within the least mean square (46) is that it has a self-correcting feature that makes it suitable to function for arbitrarily vast duration without shifting from its constraints. The mentioned gradient algorithm is subjected to cumulative series of errors and is convenient for long runs in absence of an additional error rectification procedure.

The gradient technique is now been utilized to train the Z-number weights $f_i(x)$ and $g_j(x)$. The solution $x_0$ is the function of $f_i(x)$ and $g_j(x)$. We compute $\frac{\partial J_k}{\partial x_0}$ and $\frac{\partial J_k}{\partial x_0}$ which are mentioned as

$$
\frac{\partial J_k}{\partial x_0} = \frac{\partial J_k^*}{\partial x_0} + \frac{\partial J_k^*}{\partial x_0}
$$

According to the chain rule

$$
\frac{\partial J^\alpha}{\partial x_0} = \frac{\partial J^\alpha}{\partial \hat{y}_k^\alpha} \frac{\partial \hat{y}_k^\alpha}{\partial O_f^\alpha(x)} \frac{\partial O_f^\alpha(x)}{\partial x_0} + \frac{\partial J^\alpha}{\partial \hat{y}_k^\alpha} \frac{\partial \hat{y}_k^\alpha}{\partial O_g^\alpha(x)} \frac{\partial O_g^\alpha(x)}{\partial x_0}
$$

So

$$
\frac{\partial J^\alpha}{\partial x_0} = \sum_{i=1}^{n} \left( y_k^{*\alpha} - \hat{y}_k^{\alpha} \right) a_i^\alpha f_i^\alpha + \sum_{j=1}^{m} \left( y_k^{*\alpha} - \hat{y}_k^{\alpha} \right) b_j^\alpha g_j^\alpha
$$

Or

$$
\frac{\partial J^\alpha}{\partial x_0} = \sum_{i=1}^{n} \left( y_k^{*\alpha} - \hat{y}_k^{\alpha} \right) a_i^\alpha f_i^\alpha + \sum_{j=1}^{m} \left( y_k^{*\alpha} - \hat{y}_k^{\alpha} \right) b_j^\alpha g_j^\alpha
$$

$\frac{\partial J_k}{\partial x_0}$ can be calculated the same as above.

The solution $x_0$ is upgraded as

$$
x_0(k + 1) = x_0(k) - \eta \frac{\partial J_k}{\partial x_0}
$$

Here $\eta$ is the rate of the training $\eta > 0$.

For the requirement of increasing the training process, the adding of the momentum term is mentioned as

$$
x_0(k + 1) = x_0(k) - \eta \frac{\partial J_k}{\partial x_0} + \gamma \left[ x_0(k) - x_0(k - 1) \right]
$$

Here $\gamma > 0$. After the updating of $x_0$, it is necessary to substitute it to the weights $f_i(x_0)$ and $g_j(x_0)$.

The solution related to the dual equation (27) can also be estimated by feedback neural network, as Figure 2. In this case, the inputs are the nonlinear Z-number functions $f_i(x)$ and $g_j(x)$, the concerned weights are taken to be as Z-numbers $a_i$ and $b_j$. The training error $e_k$ has been utilized here in order to update $x$. Once the nonlinear operations $f_i(x)$ and $g_j(x)$ are performed, $O_f$ and $O_g$ are considered to be similar to (44). The output related to the neural network is taken as similar to (45).
5 Simulations

In this section, we use several applications to show how to use the fuzzy equation with Z-number to design the fuzzy controller.

Example 1 (Chemistry procedure) The main intention of the chemical reaction between the poly ethylene (PE) and poly propylene (PP) is to produce a preferred substance (PS). If $x$ is considered to be the material cost, then the cost of PE is taken to be $x$ and $x^2$ is considered to be the cost of PP. The PE and PP weights which are uncertain, are sufficed by the triangle function (7). It is our requirement to generate two different types of PS. If we urge the cost in the midst is $[(360.50, 400.55, 421.37), p(0.8, 0.9, 1)] = y^*$, what can be the cost $x$? The PE weights are stated as

$$a_1 = [(2.7951, 3.35412, 3.9131), p(0.7, 0.8, 1)]$$
$$b_1 = [(1.5811, 2.1081, 2.6352), p(0.8, 0.9, 1)]$$

The PP weights are stated as

$$a_2 = [(4.8107, 5.3452, 5.8797), p(0.7, 0.875, 1)]$$
$$b_2 = [(3.9131, 4.4721, 5.0311), p(0.6, 0.8, 1)]$$

The modeling of the above mentioned relation can be carried out using the dual equation and Z-numbers

$$[(2.79, 3.354, 3.91), p(0.7, 0.8, 1)] \odot x \oplus [(4.81, 5.34, 5.8797), p(0.7, 0.875, 1)] \odot x^2$$
$$= [(1.58, 2.10, 2.6352), p(0.8, 0.9, 1)] \odot x \oplus [(3.91, 4.47, 5.0311), p(0.6, 0.8, 1)] \odot x^2$$
$$\oplus [(360.50, 400.55, 421.37), p(0.8, 0.9, 1)]$$

In this case $f_1(x) = g_1(x) = x$, $f_2(x) = g_2(x) = x^2$. The exact solution is demonstrated by

$$x^* = [(18.3712, 19.3919, 19.9022), p(0.8, 0.96, 1)]$$

We utilize feedforward (NN) and feedback (FNN) neural networks to estimate the solution $x$. The learning rate is $\eta = 0.02$. The initial state is $x(0) = [(22.66, 23.71, 24.24), p(0.8, 0.9, 1)]$. The approximation outcomes are exhibited in Table 1. The modeling errors are displayed in Figure 3.

Table 1. Neural networks approximate the Z-numbers

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x(k)$ with NN</th>
<th>$k$</th>
<th>$x(k)$ with FNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[(22.53, 23.68, 24.10), p(0.6, 0.8, 0.85)]$</td>
<td>1</td>
<td>$[(22.33, 23.38, 23.99), p(0.7, 0.8, 0.85)]$</td>
</tr>
<tr>
<td>2</td>
<td>$[(21.79, 22.83, 23.20), p(0.7, 0.8, 0.85)]$</td>
<td>2</td>
<td>$[(20.98, 22.13, 22.761), p(0.7, 0.85, 0.9)]$</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>35</td>
<td>$[(18.67, 19.71, 20.23), p(0.8, 0.92, 1)]$</td>
<td>18</td>
<td>$[(18.49, 19.51, 20.13), p(0.8, 0.92, 1)]$</td>
</tr>
<tr>
<td>36</td>
<td>$[(18.38, 19.40, 19.91), p(0.8, 0.96, 1)]$</td>
<td>19</td>
<td>$[(18.37, 19.39, 19.90), p(0.8, 0.96, 1)]$</td>
</tr>
</tbody>
</table>
We can see that both neural networks give worthy performance. We use the following to transfer the Z-numbers to fuzzy numbers,

\[ \alpha = \frac{\int x \pi_p(x) dx}{\int \pi_p(x) dx} \]

Consider \( Z = (A, p) = [(22.33, 23.38, 23.99), p(0.6, 0.8, 0.85)] \), then \( Z^\alpha = (22.31, 23.38, 23.99; 0.77) \) and so \( Z' = (\sqrt{0.77} 22.331, \sqrt{0.77} 23.384, \sqrt{0.77} 23.993) \). The results of neural networks approximation for the fuzzy numbers are displayed in Table 2.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x(k) ) with NN</th>
<th>( k )</th>
<th>( x(k) ) with FNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>35</td>
<td>(17.720, 18.710, 19.203)</td>
<td>18</td>
<td>(17.548, 18.517, 19.107)</td>
</tr>
<tr>
<td>36</td>
<td>(17.541, 18.513, 19.000)</td>
<td>19</td>
<td>(17.538, 18.509, 18.996)</td>
</tr>
</tbody>
</table>

The Z-numbers increase degree of reliability of the information. The crucial factor is that Z-information is not only the most generalized depiction of real-world uncomplicated information but also is the highest narrative power extracted from human cognition outlook as compared to fuzzy number. The comparison between the Z-number \( Z = [(18.38, 19.40, 19.91), p(0.8, 0.96, 1)] \) and fuzzy number \( (17.54, 18.51, 19.00) \) for \( k = 36 \) is shown in Figure 4. We see that the Z-number incorporates with various information and the solution of the Z-number is more accurate. The membership function for the restriction in the Z-number is \( \mu_{Az} = (18.38, 19.40, 19.91) \). It can be in probability form.
Example 2 (Heat source in insulating materials) The insulating materials center is considered to be the source of heat. The materials width are not precise and hence they suffice the trapezoidal function (8),

\[ A = [(0.131, 0.153, 0.164, 0.197), p(0.7, 0.83, 0.9)] = a_1 \]
\[ B = [(0.084, 0.105, 0.210, 0.527), p(0.8, 0.9, 1)] = a_2 \]
\[ C = [(0.096, 0.107, 0.214, 0.428), p(0.7, 0.87, 0.9)] = b_1 \]
\[ D = [(0.021, 0.032, 0.054, 0.086), p(0.8, 0.85, 0.92)] = b_2 \]

see Figure 5. The coefficient associated with conductivity materials are \( K_A = x = f_1 \), \( K_B = x\sqrt{x} = f_2 \), \( K_C = x^2 = g_1 \), \( K_D = \sqrt{x} = g_2 \), where \( x \) is considered to be as the elapsed time. The control object is to reveal the time in case the thermal resistance at the right side attains \( R = [(0.0162, 0.0293, 0.0424, 0.1241), p(0.75, 0.8, 0.9)] = y^* \). The thermal balance model is [16]:

\[ \frac{A}{K_A} \oplus \frac{B}{K_B} = \frac{C}{K_C} \oplus \frac{D}{K_D} \oplus R \]

The exact solution is \( x^* = [(2.0519, 3.0779, 4.1039, 6.1559), p(0.8, 0.95, 1)] \) [16]. The learning rate is \( \eta = 0.1 \) (NN) and \( \eta = 0.005 \) (FNN). The neural networks approximation results are
displayed in Table 3 and Table 4. The modeling errors are displayed in Figure 6.

**Table 3. Neural networks approximate the Z-numbers**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x(k)$ with NN</th>
<th>$k$</th>
<th>$x(k)$ with FNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(5.97, 6.98, 7.93, 9.98), p(0.6, 0.8, 0.85]$</td>
<td>1</td>
<td>$(5.98, 6.99, 7.97, 9.98), p(0.7, 0.85, 0.87]$</td>
</tr>
<tr>
<td>2</td>
<td>$(5.43, 6.38, 7.35, 9.302), p(0.75, 0.8, 0.9)$</td>
<td>2</td>
<td>$(5.37, 6.10, 7.12, 9.16), p(0.7, 0.85, 0.87)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>45</td>
<td>...</td>
</tr>
<tr>
<td>61</td>
<td>$(2.11, 3.170, 4.22, 6.33), p(0.8, 0.9, 1)$</td>
<td>46</td>
<td>$(2.08, 3.14, 4.14, 6.29), p(0.8, 0.96, 1)$</td>
</tr>
<tr>
<td>62</td>
<td>$(2.06, 3.08, 4.11, 6.17), p(0.8, 0.94, 1)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 4. Neural networks approximate the fuzzy numbers**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x(k)$ with NN</th>
<th>$k$</th>
<th>$x(k)$ with FNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(5.13, 5.99, 6.841, 8.576)$</td>
<td>1</td>
<td>$(5.36, 6.25, 7.14, 8.939)$</td>
</tr>
<tr>
<td>2</td>
<td>$(4.93, 5.79, 6.671, 8.440)$</td>
<td>2</td>
<td>$(4.81, 5.46, 6.37, 8.199)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>45</td>
<td>...</td>
</tr>
<tr>
<td>61</td>
<td>$(2.00, 3.00, 4.007, 6.008)$</td>
<td>45</td>
<td>$(1.99, 3.00, 3.95, 6.004)$</td>
</tr>
<tr>
<td>62</td>
<td>$(1.96, 2.934, 3.915, 5.870)$</td>
<td>46</td>
<td>$(1.95, 2.93, 3.90, 5.864)$</td>
</tr>
</tbody>
</table>

**Example 3 (Water channel system)** The pipe $d_1$ which is carrying water is subdivided into three different pipes $d_2, d_3, d_4$, refer Figure 7. The areas of the pipes are uncertain, they suffice the trapezoidal function (8),

$$A_1 = [(0.421, 0.632, 0.737, 0.843), p(0.75, 0.9, 1)]$$
$$A_2 = [(0.052, 0.104, 0.209, 0.419), p(0.8, 0.91, 1)]$$
$$A_3 = [(0.031, 0.084, 0.105, 0.210), p(0.8, 0.9, 0.95)]$$

The velocities of water flowing through the pipes are controlled with the help of valves parameter $x$, $v_1 = x^3$, $v_2 = \frac{x^2}{2}$, $v_3 = x [34]$. The flow in pipe $d_4$ is initiated using the control
object which is represented by

\[ Q = [(11.478, 40.890, 93.332, 293.056), p(0.8, 0.87, 0.95)] \]

We need to find the valve control parameter \( x \). By mass balance

\[ A_1v_1 = A_2v_2 \oplus A_3v_3 \oplus Q \]

The exact solution is demonstrated by \( x = [(3.127, 4.170, 5.212, 7.298), p(0.8, 0.92, 1)] \) [34]. The learning rate of NN is \( \eta = 0.08 \). The neural networks approximation results are displayed in Table 5 and Table 6.
Table 5. Neural networks approximate the Z-numbers

<table>
<thead>
<tr>
<th>k</th>
<th>x (k) with NN</th>
<th>k</th>
<th>x (k) with FNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[(5.75, 6.77, 7.74, 9.76), p(0.6, 0.8, 0.85)]</td>
<td>1</td>
<td>[(5.87, 6.88, 7.86, 9.877), p(0.7, 0.81, 0.85)]</td>
</tr>
<tr>
<td>2</td>
<td>[(5.32, 6.26, 7.13, 9.20), p(0.7, 0.8, 0.87)]</td>
<td>2</td>
<td>[(5.15, 6.00, 7.00, 9.002), p(0.7, 0.85, 0.9)]</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>55</td>
<td>[(3.14, 4.19, 5.22, 7.32), p(0.8, 0.9, 1)]</td>
<td>20</td>
<td>[(3.13, 4.18, 5.22, 7.312), p(0.85, 0.9, 1)]</td>
</tr>
<tr>
<td>56</td>
<td>[(3.13, 4.18, 5.22, 7.31), p(0.8, 0.93, 1)]</td>
<td>21</td>
<td>[(3.13, 4.178, 5.21, 7.305), p(0.8, 0.92, 1)]</td>
</tr>
</tbody>
</table>

Table 6. Neural networks approximate the fuzzy numbers

<table>
<thead>
<tr>
<th>k</th>
<th>x (k) with NN</th>
<th>k</th>
<th>x (k) with FNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4.94, 5.81, 6.65, 8.38)</td>
<td>1</td>
<td>(5.18, 6.07, 6.94, 8.71)</td>
</tr>
<tr>
<td>2</td>
<td>(4.72, 5.54, 6.31, 8.15)</td>
<td>2</td>
<td>(4.63, 5.39, 6.28, 8.08)</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>55</td>
<td>(2.98, 3.97, 4.96, 6.94)</td>
<td>20</td>
<td>(2.97, 3.97, 4.95, 6.93)</td>
</tr>
<tr>
<td>56</td>
<td>(2.97, 3.97, 4.95, 6.94)</td>
<td>21</td>
<td>(2.97, 3.96, 4.95, 6.93)</td>
</tr>
</tbody>
</table>

FNN is much faster and more robust compared with NN. After converting the Z-numbers to fuzzy numbers, it is possible to extract the fuzzy rules.

6 Conclusions

In this paper, we use dual fuzzy equations, whose coefficients are Z-numbers, to model uncertain nonlinear systems. Then we give the relation between the solution of the fuzzy equations and the controllers. The controllability of the fuzzy system is proposed. Two types of neural networks are applied to approximate the solutions of the fuzzy equations. Modified gradient descent algorithms are used to train the neural networks. The novel methods are validated by several benchmark examples. The future works are the application of the mentioned methodologies for fuzzy differential equations.

References


